ME 261: Numerical Analysis

Lecture-6: Root Finding

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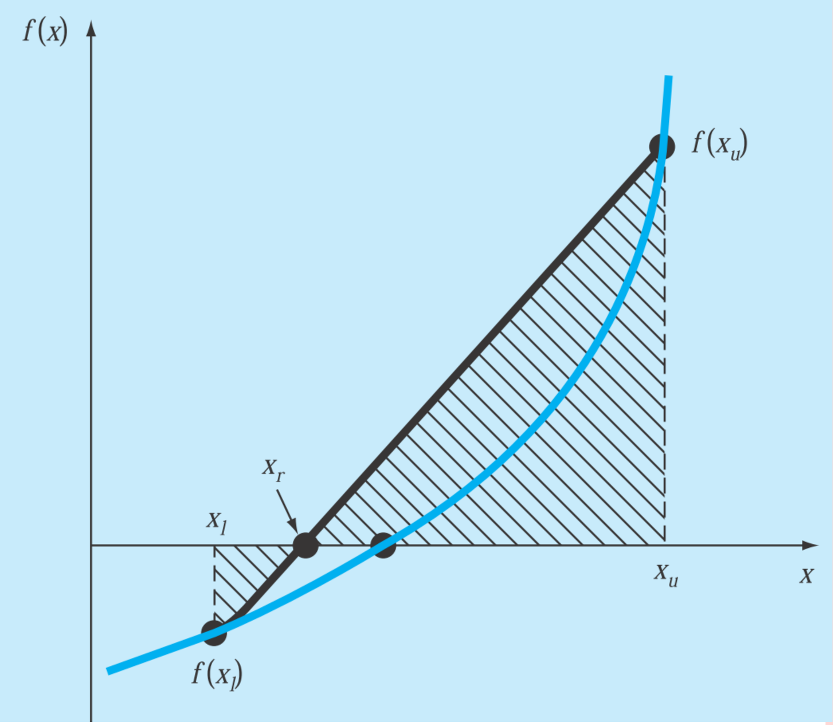
False Position Method for Finding Root

- A shortcoming of the bisection method is that, in dividing the interval from x_l to x_u into equal halves, **no account is taken of the magnitudes of $f(x_l)$ and $f(x_u)$**.

- For example, if **$f(x_l)$ is much closer to zero** than $f(x_u)$, it is likely that the **root is closer to x_l** than to x_u

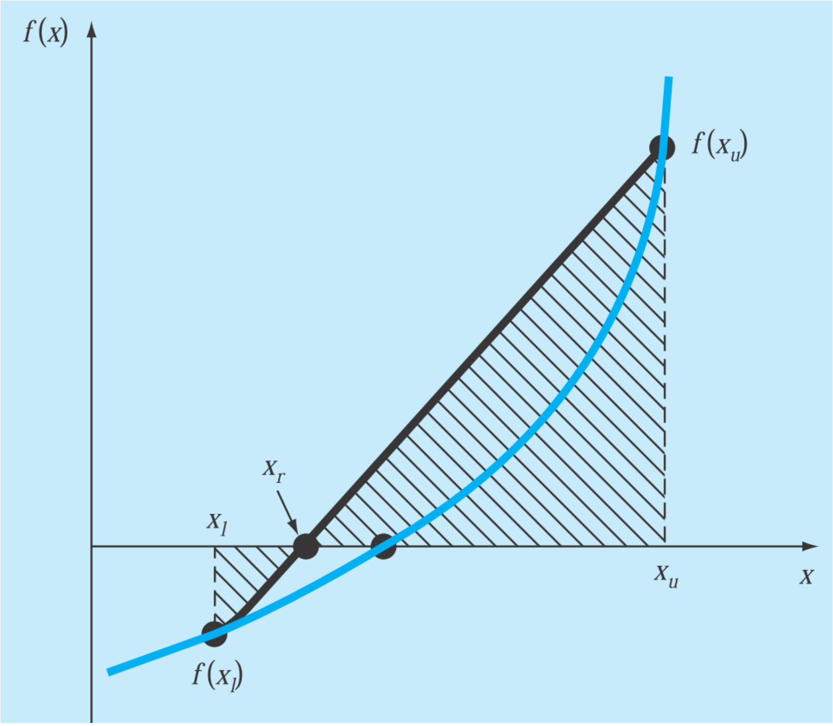
- An alternative method that exploits this graphical insight is to join $f(x_l)$ and $f(x_u)$ by **a straight line**. The intersection of this line with the x axis represents an **improved estimate of the root**.



False Position Method for Finding Root

- The fact that the replacement of the curve by a straight line gives a **"false position"** of the root is the origin of the name, method of false position, or in Latin, regula falsi.

- It is also called the **linear interpolation method**.



Presumption: $f(x_l)$ is much closer to zero than $f(x_u)$

$$\frac{f(x_l)}{x_r - x_l} = \frac{f(x_u)}{x_r - x_u}$$

$$f(x_l)(x_r - x_u) = f(x_u)(x_r - x_l)$$

Collect terms and rearrange:

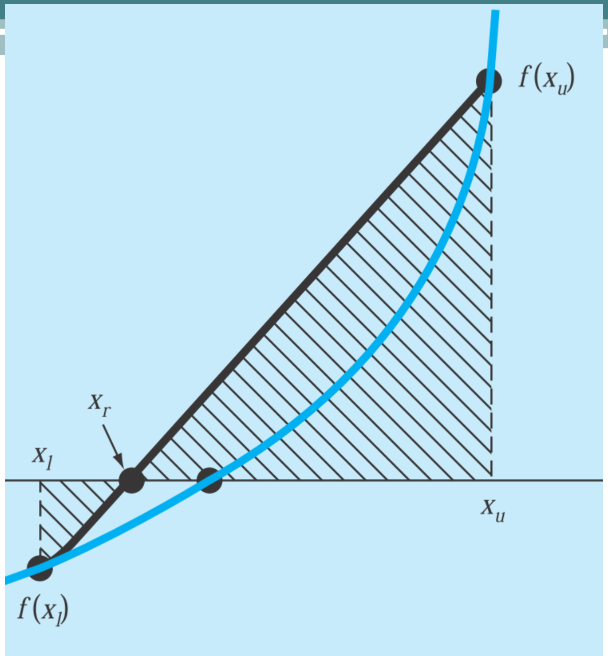
$$x_r\,[\,f(x_l) - f(x_u)] = x_u\, f(x_l) - x_l\, f(x_u)$$

$$\boxed{x_r = \frac{x_u\, f(x_l) - x_l\, f(x_u)}{f(x_l) - f(x_u)}}$$

$$x_r = \frac{x_u\, f(x_l)}{f(x_l) - f(x_u)} - \frac{x_l\, f(x_u)}{f(x_l) - f(x_u)}$$

$$x_r = x_u + \frac{x_u\, f(x_l)}{f(x_l) - f(x_u)} - x_u - \frac{x_l\, f(x_u)}{f(x_l) - f(x_u)}$$

$$x_r = x_u + \frac{x_u\, f(x_u)}{f(x_l) - f(x_u)} - \frac{x_l\, f(x_u)}{f(x_l) - f(x_u)}$$



$$\boxed{x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}}$$

$f(x_l)$ is much closer to zero than $f(x_u)$

Solve Turtle Motion Eqn.

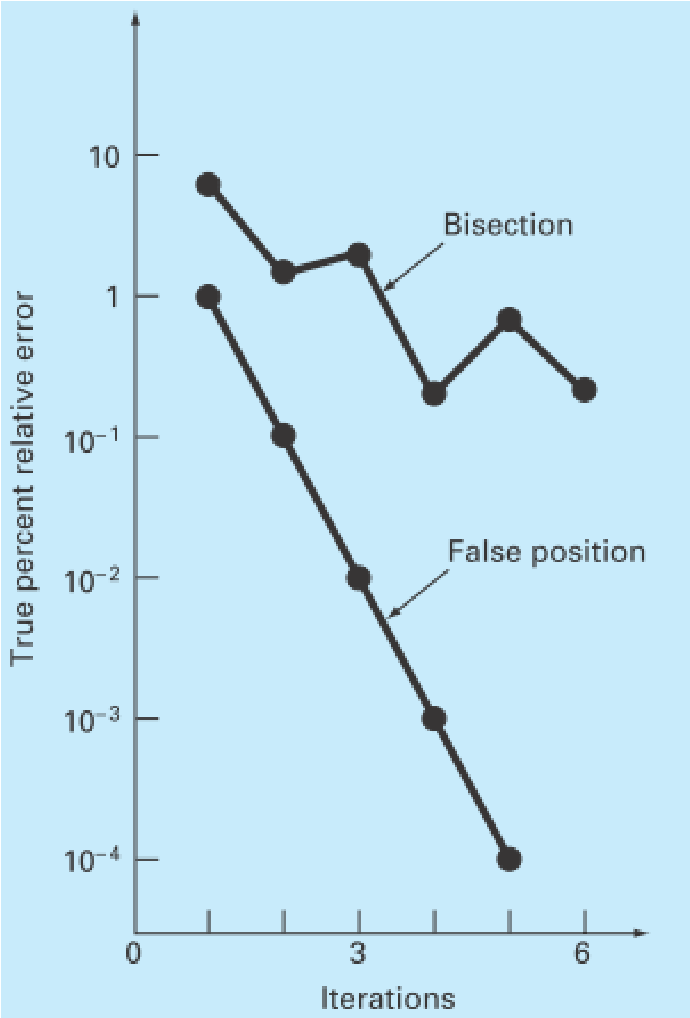
$$f(c) = \frac{49.05}{c}\left(1 - e^{-1.8c}\right) - 10 = 0$$

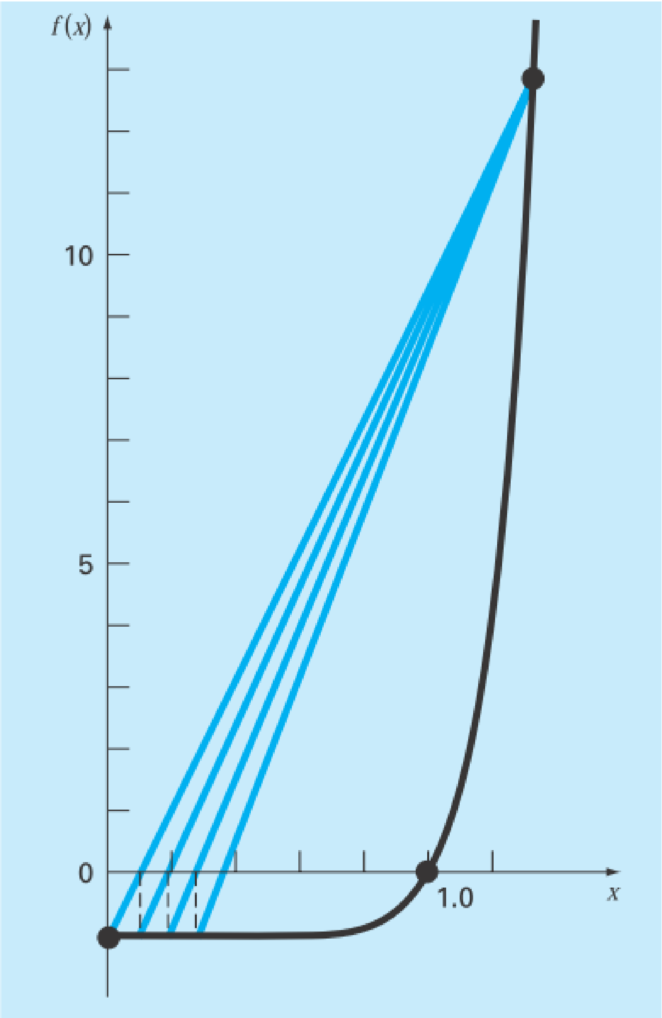
Root Bracket $[4,6]$; *Approx Error* $\leq 0.5\%$;

In general error in False position method decreases much faster than bisection method.

$$\boxed{x_r = \frac{x_u\ f(x_l) - x_l\ f(x_u)}{f(x_l) - f(x_u)}}$$

it	xl	xu	xr	f(xl)	f(xu)	f(xr)	Approx. error (%)
1	4	6	5.104984	2.253345	-1.82517	-0.39272	_
2	4	5.10498	4.940984	2.253345	-0.39272	-0.07419	3.21255
3	4	4.94098	4.910989	2.253345	-0.07418	-0.01364	0.607051
4	4	4.91099	4.905507	2.253345	-0.01364	-0.0025	0.111633
5	4	4.90551	4.904506	2.253345	-0.0025	-0.00046	0.020419
6	4	4.90451	4.904323	2.253345	-0.00047	-8.5E-05	0.003727
7	4	4.90432	4.904288	2.253345	-8E-05	-1.5E-05	0.000708
8	4	4.90464	4.904347	2.253345	-0.00073	-0.00013	0.001193
9	4	4.90435	4.904293	2.253345	-0.00014	-2.6E-05	0.001081
10	4	4.90429	4.904283	2.253345	-1.9E-05	-3.4E-06	0.000224
11	4	4.90428	4.904281	2.253345	1.75E-06	3.19E-07	3.73E-05





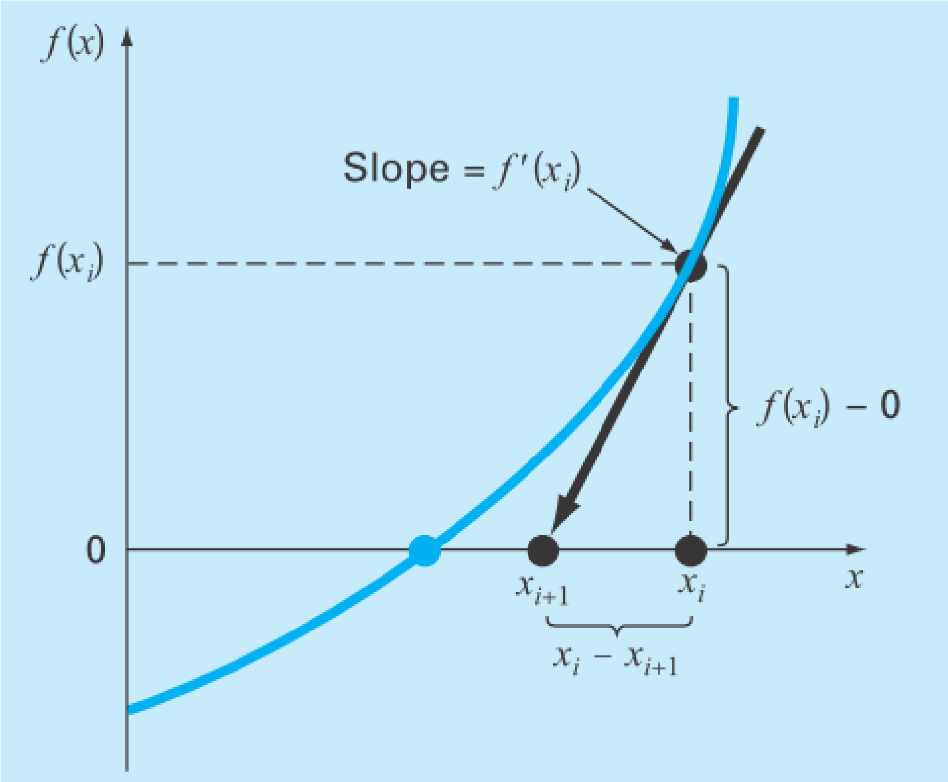
Error propagation:
False position vs. Bisection Method.

Slow Convergence in False Position Method, in in some cases, Why??
$f(x_u)$ is much closer to zero than $f(x_l)$

Newton Raphson Method for Finding Root

- One of the most widely used methods of solving non-linear equations.
- Single initial guess of root is required.
- Also known as Newton's method.

If the initial guess is at x_i, a tangent can be extended from point $[x_i, f(x_i)]$. The point where the tangent crosses the x-axis usually represents an improved estimate of the root.



Graphical presentation of iterative progress using Newton's method for finding the root of non-linear equation, f(x)=0.

Using Taylor series expansion for single variable:

$$f(x_{i+1}) = f(x_i) + (x_{i+1} - x_i) f'(x_i) + \frac{(x_{i+1} - x_i)^2}{2!} f''(x_i) + \ldots\ \ldots\ \ldots + \text{H.O.T.}$$

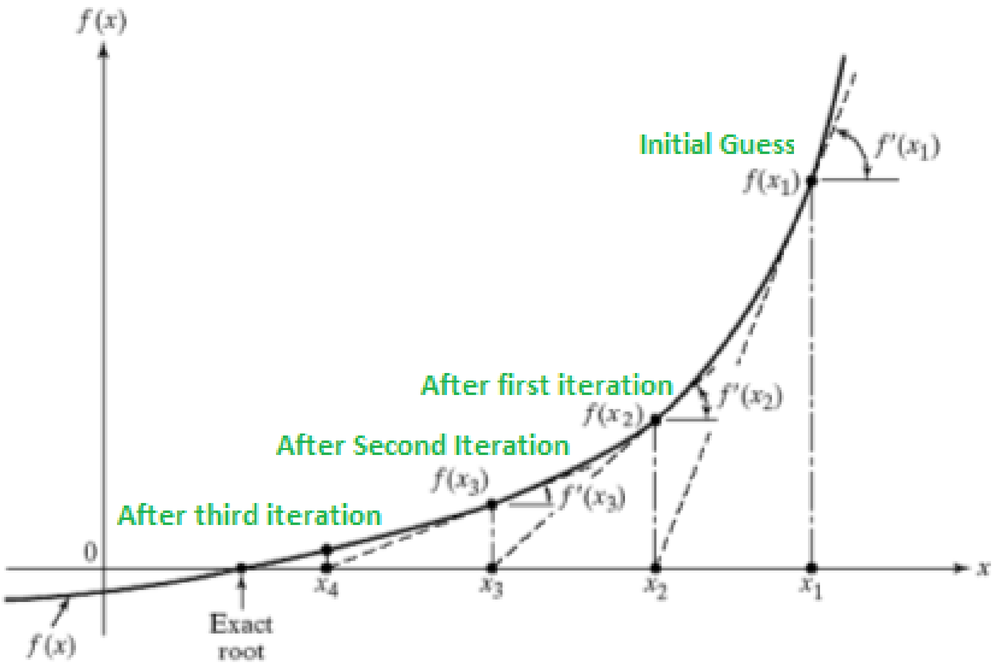
$$\Rightarrow f(x_{i+1}) = f(x_i) + (x_{i+1} - x_i) f'(x_i) \quad \text{[Truncating the series after 1st derivative]}$$

If x_{i+1} is an approximation of a **root**, then $f(x_{i+1}) = 0$

$$\Rightarrow 0 = f(x_i) + (x_{i+1} - x_i) f'(x_i)$$

$$\boxed{\Rightarrow x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

$i = 1,2,3,4, \ldots\ \ldots\ \ldots$



Two functions namely f(x_i) and its 1st derivative, f/(x_i) are needed to estimate the root at each iterations.

The expression for the estimation of root for **Newton Raphson Method** is:

$$\boxed{\Rightarrow\ x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

In the form of

$$x_{i+1} = g(x_i) \qquad : \ \text{F.P.I.}$$

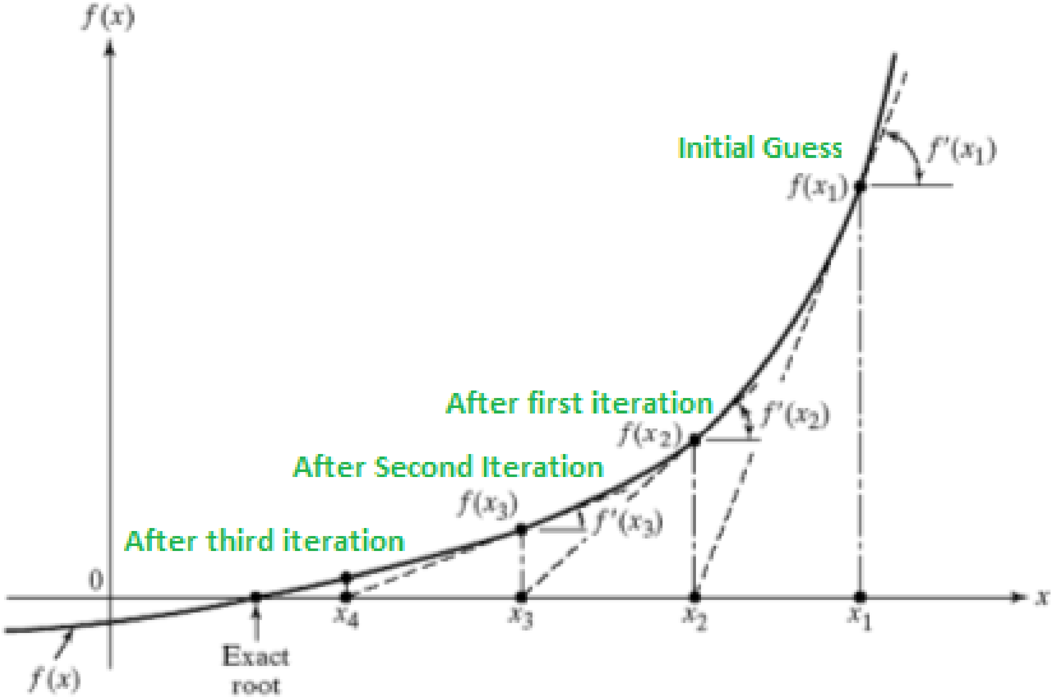
It is known that the convergence criteria for F.P.I. is-

$$\left|g'(x_i)\right| < 1$$

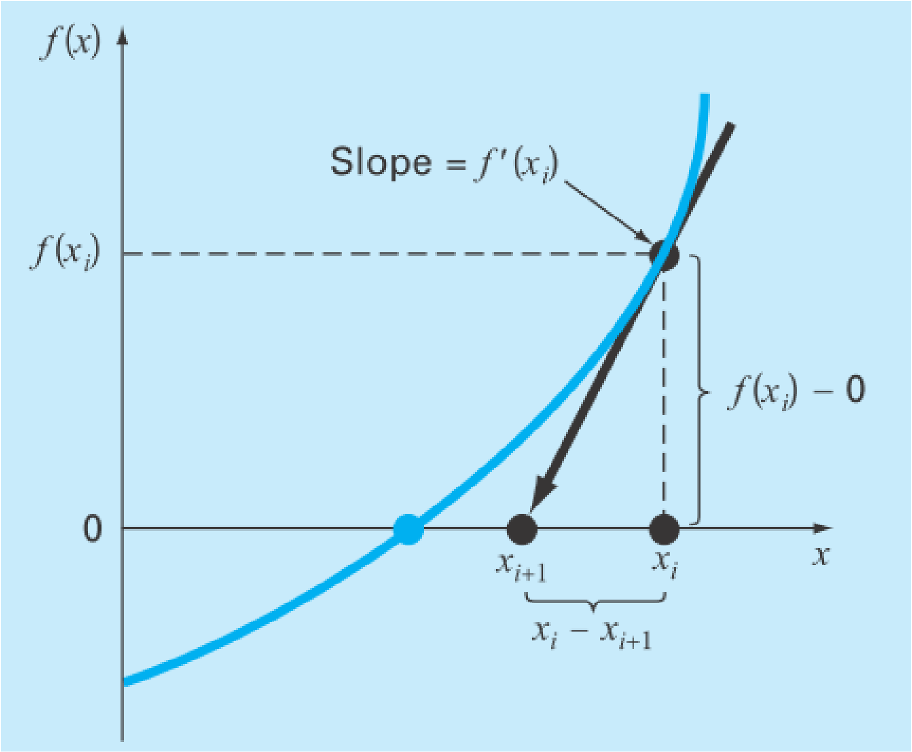
$$\Rightarrow \left|\frac{d}{dx}\left(x_i - \frac{f(x_i)}{f'(x_i)}\right)\right| < 1$$

$$\boxed{\Rightarrow \left|f(x_i)\,f''(x_i)\right| < \left[f'(x_i)\right]^2}$$

Convergence criteria for Newton's method



Newton Raphson Method for Finding Root



$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$x_i - x_{i+1} = \frac{f(x_i)}{f'(x_i)}$$

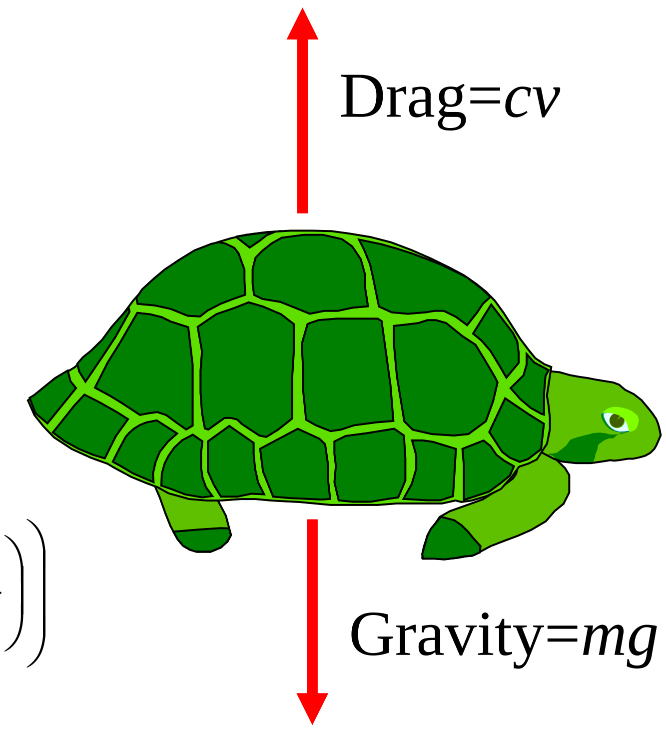
$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad (A)$$

To find a root using Newton-Raphson method, do the following:

1) Let the initial guess be ***x_i***
2) Find ***x_{i+1}*** by using **Eq. (A)**
3) Let $x_i = x_{i+1}$ **repeat steps 2 and 3** until you feel your root is **accurate enough**.

Example: Solve the turtle problem by using Newton's method. Start with, $c_i = 3.0$.

$$f(c) = \frac{50}{c}\left(1 - e^{-\frac{9c}{5}}\right) - 10$$



$$f(c) = \frac{50}{c}\left(1 - e^{-\frac{9c}{5}}\right) - 10$$

$$f(c) = \frac{50}{c} - \frac{50}{c} e^{-\frac{9c}{5}} - 10$$

$$f'(c) = -\frac{50}{c^2} - 50\left(\left(-\frac{1}{c^2} e^{-\frac{9c}{5}}\right) + \left(\frac{1}{c}\right) e^{-\left(\frac{9}{5}\right)c}\left(-\frac{9}{5}\right)\right)$$

$$= -\frac{50}{c^2} + \frac{50}{c^2} e^{-\left(\frac{9}{5}\right)c} + 50\left(\frac{9}{5}\right) e^{-\left(\frac{9}{5}\right)c}\left(\frac{1}{c}\right)$$

$$= \frac{50}{c}\left[\left(\frac{1}{c} + \frac{9}{5}\right) e^{-\left(\frac{9}{5}\right)c} - \frac{1}{c}\right]$$

Newton Raphson Method for Finding Root

$$f(c)=\frac{50}{c}\left(1-e^{-\frac{9c}{5}}\right)-10$$

$$\boxed{\Rightarrow\; x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}}$$

$$x_i=3.0$$

$$f'(c)=\frac{50}{c}\left[\left(\frac{1}{c}+\frac{9}{5}\right)e^{-\left(\frac{9}{5}\right)c}-\frac{1}{c}\right]$$

$$\varepsilon_a=\left|\frac{x_{i+1}-x_i}{x_{i+1}}\right|100\%$$

It.	x_i	$f(x_i)$	$f'(x_i)$	x_r	Error(%)
1	3.000000	6.591390	-5.394966	4.221767	28.9397
2	4.221767	1.837451	-2.793232	4.879589	13.4811
3	4.879589	0.245193	-2.096774	4.996528	2.340392
4	4.996528	0.005707	-2.000295	4.999381	0.057066
5	4.999381	0.000003	-1.998024	4.999382	3.24E-05
6	4.999382	0.000000	-1.998023	4.999382	1.05E-11
7	4.999382	0.000000	-1.998023	4.999382	0

Error Propagation of Newton's Method

Using Taylor series expansion for single variable:

$$f(x_{i+1}) = f(x_i) + (x_{i+1} - x_i) f'(x_i) + \frac{(x_{i+1} - x_i)^2}{2!} f''(x_i) + \ldots\ \ldots\ \ldots + \text{H.O.T.}$$

$$\Rightarrow f(x_{i+1}) = f(x_i) + (x_{i+1} - x_i) f'(x_i) \qquad \text{[Truncating the series after 1st derivative]}$$

If x_{i+1} is an **approximation of a root** of f(x), then $f(x_{i+1}) = 0$

$$0 = f(x_i) + (x_{i+1} - x_i) f'(x_i) \qquad (i)$$

Now, if x_r is the **exact/real root** of f(x), then $f(x_r) = 0$ and Taylor series can be expanded to x_r from x_i accordingly-

$$f(x_r) = f(x_i) + (x_r - x_i) f'(x_i) + \frac{(x_r - x_i)^2}{2!} f''(x_i) + \ldots\ \ldots\ \ldots + \text{H.O.T.}$$

$$\Rightarrow f(x_r) = f(x_i) + (x_r - x_i) f'(x_i) + \frac{(x_r - x_i)^2}{2!} f''(x_i) \quad \text{[Truncating after 2nd derivative]}$$

$$\Rightarrow 0 = f(x_i) + (x_r - x_i) f'(x_i) + \frac{(x_r - x_i)^2}{2!} f''(x_i) \qquad (ii)$$

Error Propagation of Newton's Method

Now eq(ii)-eq(i) gives:

$$0 = (x_r - x_i - x_{i+1} + x_i) f'(x_i) + \frac{(x_r - x_i)^2}{2!} f''(x_i)$$

$$\Rightarrow 0 = (x_r - x_{i+1}) f'(x_i) + \frac{(x_r - x_i)^2}{2!} f''(x_i)$$

Absolute Error at iteration $i+1$, $E_{i+1} = x_r - x_{i+1}$

Absolute Error at iteration i, $E_i = x_r - x_i$

$$0 \cong E_{i+1} f'(x_i) + \frac{E_i^{\,2}}{2!} f''(x_i)$$

$$\Rightarrow E_{i+1} \cong -\frac{f''(x_i)}{2 f'(x_i)} E_i^{\,2}$$

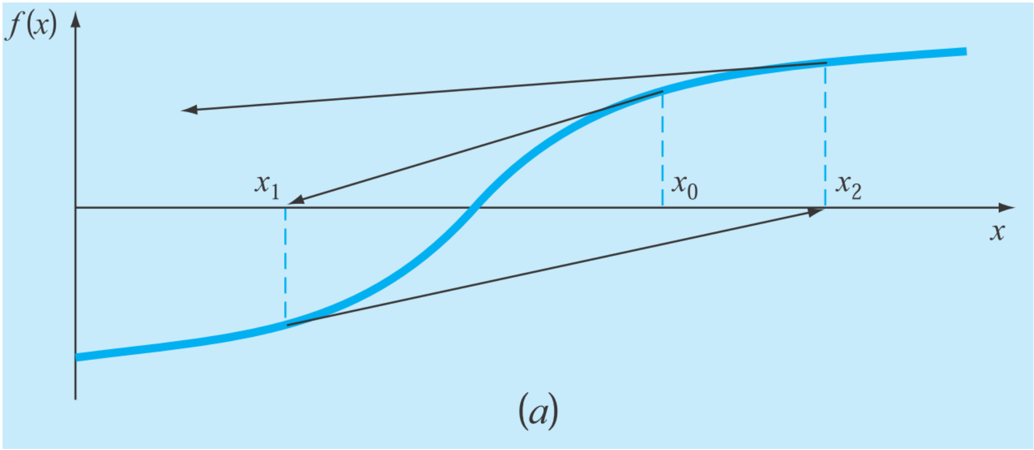
$$\boxed{\Rightarrow E_{i+1} \cong \left| \frac{f''(x_i)}{2 f'(x_i)} E_i^{\,2} \right|}$$

$$\boxed{\Rightarrow E_{i+1} \propto E_i^{\,2}} \qquad \boxed{\Rightarrow E_{i+1} \cong O(E_i^{\,2})}$$

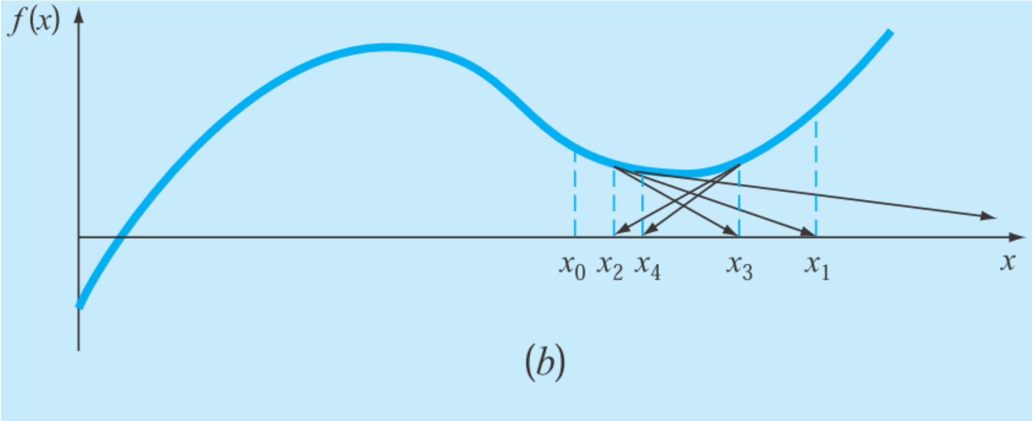
Current error is roughly proportional to the square of the previous error. Thus, it is said **Newton's method converges quadratically**.

This means that the number of correct decimal places approximately doubles with each iteration

Pitfalls of Newton's Method

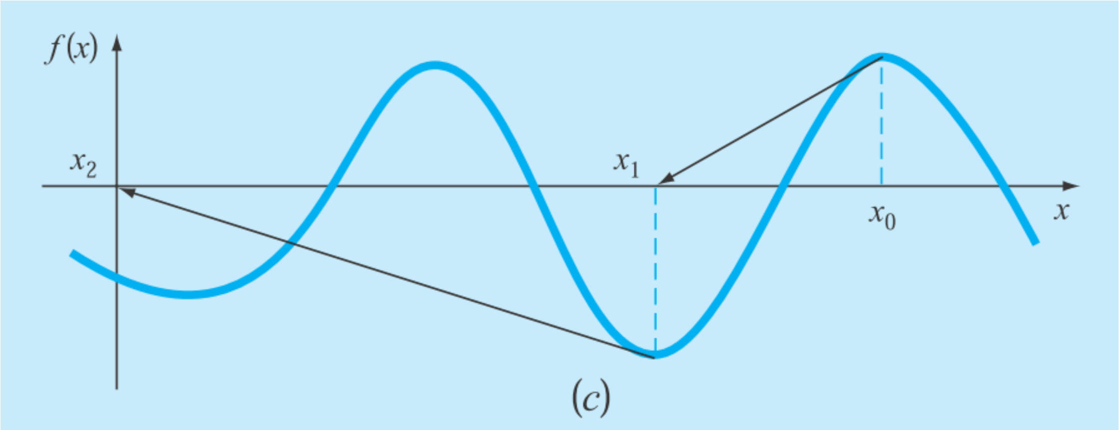


Case-A: Case where an inflection point [Decreasing slope; $f^{//}(x)=0$] occurs in the vicinity of a root. Iterations begins at x_0 progressively diverge from the root.

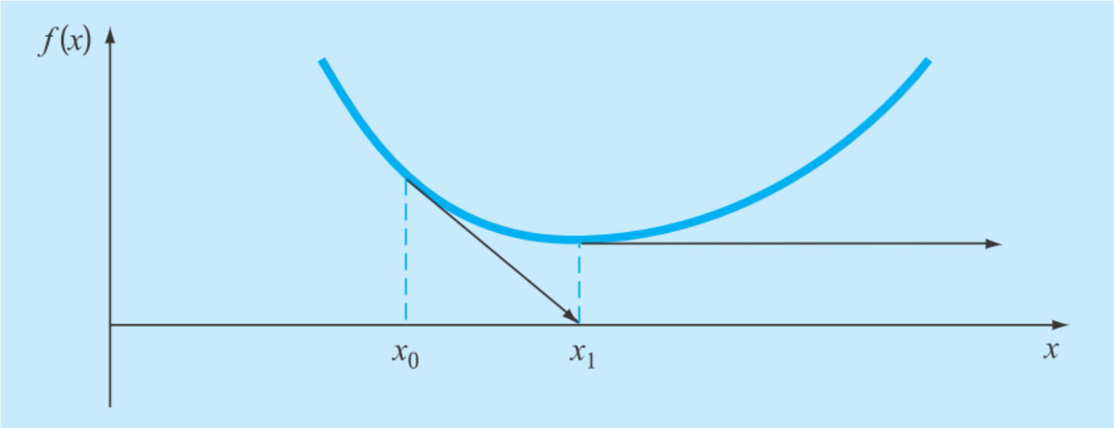


Case-B: Case shows the tendency of the Newton's method to oscillate around a local maximum or minimum.

Pitfalls of Newton's Method



Case-C: Case shows how an initial guess that is close to one root can jump to a location several roots away. This is due to near zero slope.



Case-D: Case shows an encounter of zero slope ($f'(x)=0$). In this case the solution shoots off horizontally and never hits the x-axis. (diverge- a disaster)